Degradation Rate Models for Failure Time and Survival Data [1]

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1 Introduction

Consider an experiment where a component or some material is put on test at time 0. The material degrades over time and when the amount of degradation reaches a critical boundary ω, failure occurs. Let T denote the time when failure occurs and let $F_0(t) = Pr(T \leq t)$ denote the failure time distribution. The object is to construct a model for $F_0\ (t)$ and to estimate the parameters in this model.

Bhattacharyya and Fries [2] consider a model where the degradation of the material over time is represented by a Wiener process with positive drift. The failure time of the material is represented as the first time the Wiener process crosses a critical boundary $\omega > 0$. This failure time has an inverse Gaussian distribution. Doksum and Hóyland [7] extend this model to variable stress accelerated testing, that is to cases where the stress on the material is increased to obtain early failures. They show how the distribution of the failure times of material subject to normal stress is related to the distribution of the failure times of material subjected to accelerated stress and estimate the parameters in the failure time distribution corresponding to normal stress. In this paper an extension of this stressed degradation Wiener process model and its possible use in survival analysis is discussed. More precisely, (Normand and Cleary [10]), it is suggested that the immune system of a person infected with the HIV virus decays or degrades over time and that this degradation can be measured by T4 and T8 cell counts as well as the T4/T8 ratio. Moreover, the degradation rate may increase with increased stress, or it may become (temporarily) decreasing after treatment. The proposed Wiener process model corresponds to assuming that changes in stress correspond to a transformation of the time t of the degradation Wiener process $V(t)$. Thus an HIV infection is regarded as speeding up degradation by changing the Wiener process $V(t)$ to $W(t) = V(\tau(t))$, where $\tau(t)$ is some unknown positive function which is to be estimated.

In the case of stressed material in reliability engineering, the stress is known at each time point and failure time data corresponding to accelerated stress is available. The failure time is modeled as being the first time the degradation

[1]From talk given in the Biostatistics Colloquium, Harvard University, February 4, 1991.

process $V(\tau(t))$ crosses a critical boundary ω. The object is to estimate the parameters of the failure time distribution of material subject to normal stress using the observed failure times of data subject to accelerated stress. In the case of HIV infected individuals, typically only the values of the degradation Wiener process $V(\tau(t))$ at certain time points is known and the object is to estimate $\tau(t)$. An interesting case not discussed in this paper is where both degradation process values and crossing (failure) times are observable.

The function $\tau(t)$ has an interpretation as a cumulative (or integrated) degradation rate (CDR). In the case of covariate information (e.g., age, gender), the CDR can be modeled as a function of a covariate vector and a parameter vector, e.g.

$$\tau(t) = \tau_0(t) \exp\{\sum_{j=1}^{p} c_{ij}\theta_j\},$$

where $\tau_0(t)$ is a baseline degradation rate (the degradation rate when the covariates have no effect), $\{c_{ij}, j = 1, \ldots, p\}$ are covariate values for individual i and $\theta_1, \ldots, \theta_p$ are degradation model regression parameters.

2 The Wiener process model in reliability engineering and AIDS research

2.1 *Reliability engineering*

Degradation of material (equipment, components, cables, insulation, etc.) to a large extent determines the failure time of the material. Thus it makes sense to model the distribution of failure time in terms of the process of degradation. One way this can be done is to construct a model where failure occurs when degradation has built up to a point where it reaches a critical boundary ω. See Figure 1.

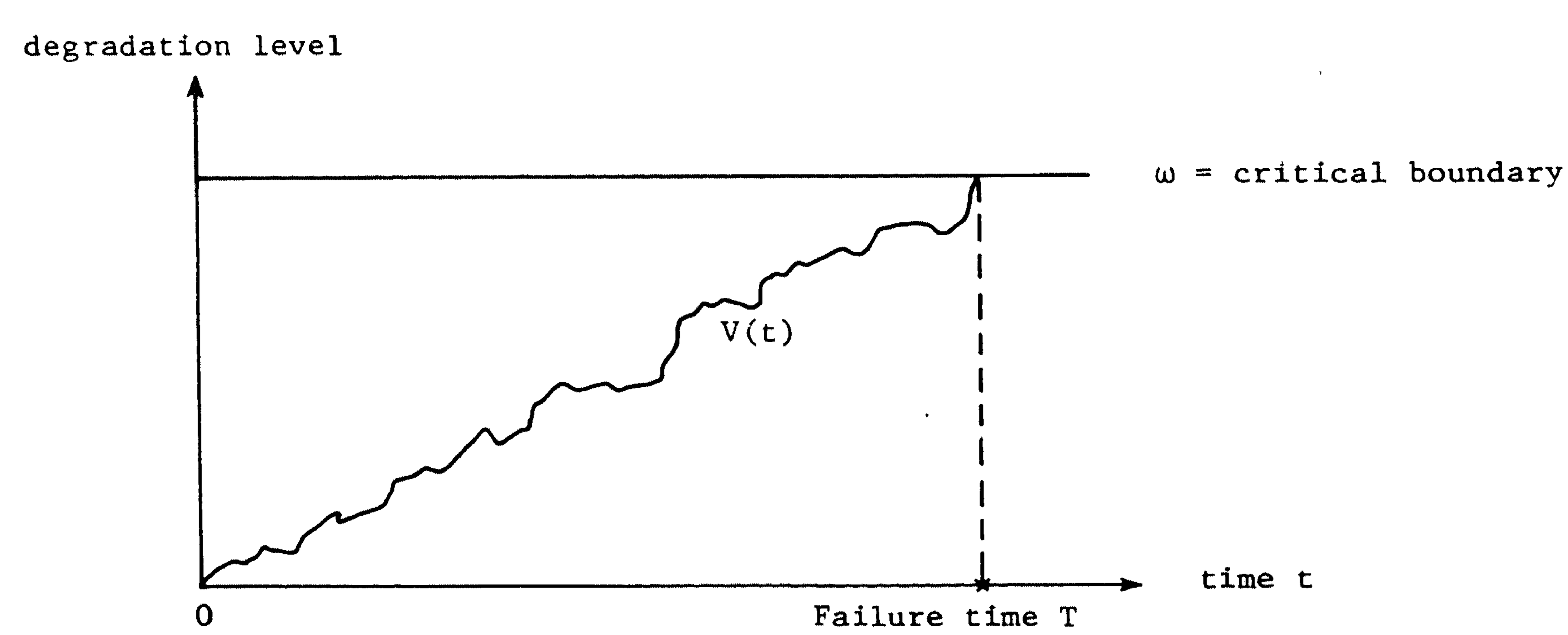

FIGURE 1. The degradation process $V(t)$ and time to failure T.

Here the degradation process $V(t)$, $t > 0$, is taken to be a Wiener process with drift parameter $\eta > 0$ and diffusion constant $\sigma^2 > 0$, that is $V(t)$ has independent increments, $V(0) = 0$, $E[V(t)] = \eta t$, $t > 0$, and $\mathrm{Var}[V(t_2) - V(t_1)] = (t_2 - t_1)\sigma^2$, $0 \le t_1 < t_2$. The time T to failure of the material is modeled as the first time the degradation process $V(t)$ crosses the critical boundary $\omega > 0$. Here is a well-known result (e.g., Chhikara and Folks [3]).

THEOREM 2.1. (Schrodinger (1915)). *T has density*

$$f_0(t) = \sqrt{\frac{\lambda}{2\pi t^3}}\ \exp\{-\frac{\lambda}{2\mu^2}\frac{(t-\mu)^2}{t}\},\ t > 0,\ \mu > 0,\ \lambda > 0,$$

where $\mu = \omega/\eta$ =(boundary)/(drift parameter) and $\lambda = \omega^2/\sigma^2$.

It can also be shown that $\mu = E(T)$, $\lambda = \mathrm{Var}(t)/\mu^2$ and that T has distribution function

$$F_0(t) = \Phi\left(\frac{\sqrt{\lambda}}{\mu}\sqrt{t} - \sqrt{\lambda}\frac{1}{\sqrt{t}}\right) + \Phi\left(-\frac{\sqrt{\lambda}}{\mu}\sqrt{t} - \sqrt{\lambda}\frac{1}{\sqrt{t}}\right)e^{\frac{2\lambda}{\mu}}.$$

The distribution F_0 is called the inverse Gaussian distribution with parameters μ and λ. It is an exponential family distribution and when it is used to model the distribution of failure time data the results are very elegant, e.g., Chhikara and Folks [3]. In particular, maximum likelihood estimates can be obtained explicitly and the distribution theory of these estimates can be given in terms of chi-square and F distributions.

One criticism of the above Wiener process model is that the degradation process $V(t)$ is not monotonely increasing in time t. One partial answer is that if maintenance is performed when a weakness is observed in the material, this would lead to improvement and thus $V(t)$ could be decreasing at some time points.

Step-stress tests

Next we consider step-stress accelerated life tests. Here each unit being tested under increasingly severe stress in order to obtain more failure data which can be used to estimate the failure time distribution $F_0(t) = Pr(T \le t)$.

More precisely, each unit is tested at a certain constant stress level s_0 (usually normal stress) in an initial time interval $[t_0, t_1)$, $t_0 \ge 0$. If the unit has not failed in the first interval, then at time t_1, the stress is increased to s_1 and kept at that level throughout the interval $[t_1, t_2)$. If the unit has not failed by time t_2, the stress is increased to s_2, and so on until the k^{th} time interval $[t_{k-1}, t_k)$ where the stress is s_{k-1}. By taking $t_k = \infty$, we will have failure with probability one. We model the resulting degradation process $W(t)$ as follows: In the first time interval $W(t)$ equals the Wiener process $V(t_0 + \alpha_0[t - t_0])$ where $V(t)$ is the original degradation Wiener process and α_0 depends on s_0. If s_0 is the normal stress level, then $\alpha_0 = 1$ and $W(t) = V(t)$ in the first time interval $[t_0, t_1)$. We introduce the notation

$$W_0(t) = V(t_0 + \alpha[t - t_0]),\ t \ge t_0.$$

Thus in the first interval the degradation process is

$$W(t) = W_0(t) = V(t_0 + \alpha_0[t - t_0]), \quad t \in [t_0, t_1).$$

Note, that if s_0 is larger than the normal stress level, then $\alpha_0 > 1$ and the time index t has been accelerated by the constant α_0 (depending on s_0).

In the second time interval $[t_1, t_2)$, the degradation process has the same distribution as in the first, except the time index t has been accelerated by the constant α_1 depending on s_1. We define

$$W_1(t) = W_0(t_1 + \alpha_1[t - t_1]), \quad t \geq t_1.$$

Then, for the second time interval, the level of degradation is

$$W(t) = W_1(t) = W_0(t_1 + \alpha_1[t - t_1]), \quad t \in [t_1, t_2).$$

Next, for the third interval $[t_2, t_3)$ with stress level s_2, the same type of modeling as for the previous intervals leads to

$$W(t) = W_2(t) = W_1(t_2 + \alpha_2[t - t_2]), \quad t \in [t_2, t_3),$$

where we have defined

$$W_2(t) = W_1(t_2 + \alpha_2[t - t_2]), \quad t \geq t_2.$$

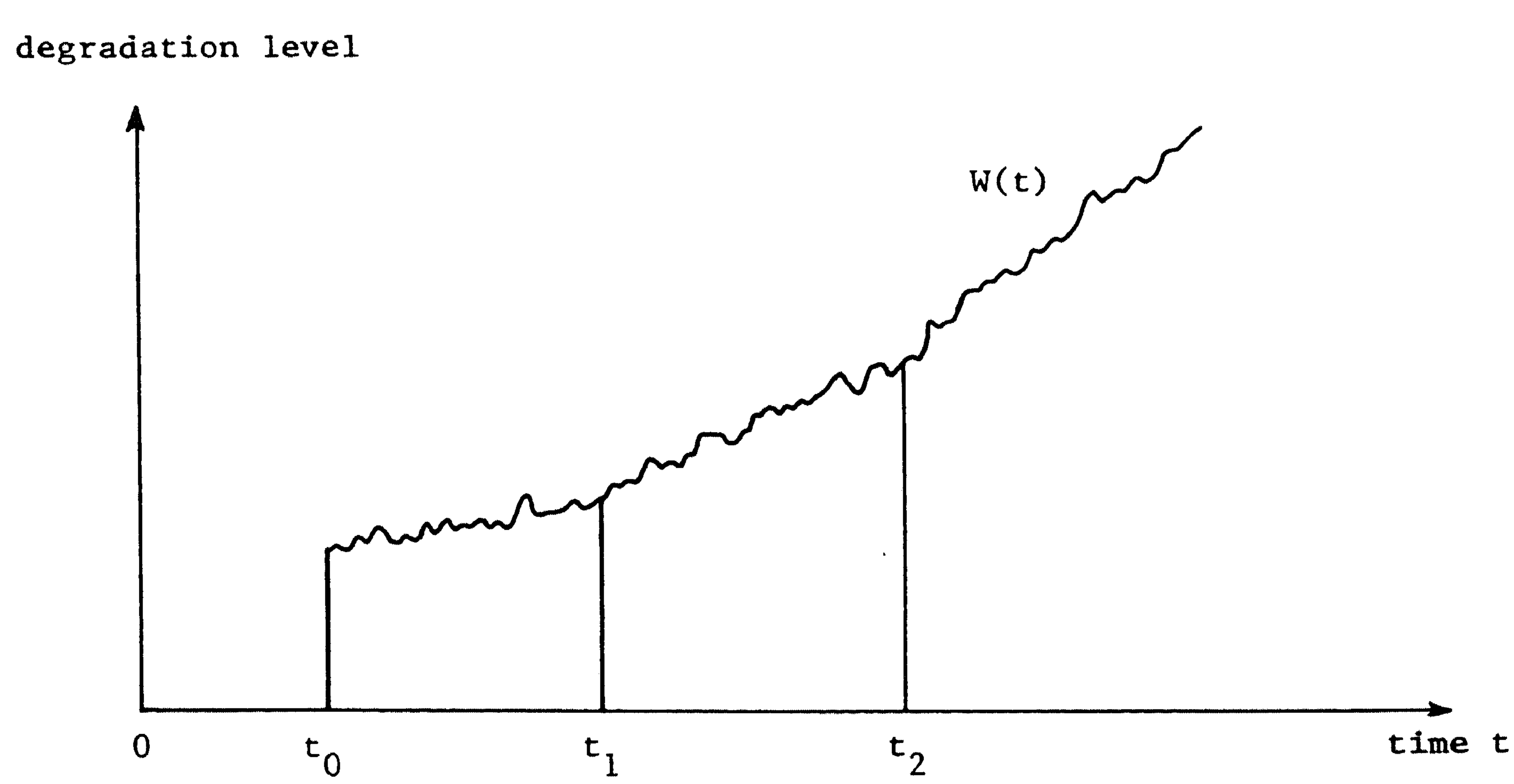

FIGURE 2. Step-stress degradation level $W(t)$.

In general, we define

$$W_i(t) = W_{i-1}(t_i + \alpha_i[t - t_i]), \;\; t \geq t_i$$

and we model the degradation level in the $(i+1)$st time interval as having the same distribution as a time accelerated version of the process in the i^{th} interval. That is

$$W_i(t) = W_{i-1}(t_i + \alpha_i[t - t_i]), \; t \in [t_i, t_{i+1}], \; t_i \leq t_{i+1}, \; i = 0, \ldots, k-1.$$

See Figure 2.

Note that $W_i(t)$ can be expressed in terms of $W_{i-1}(t)$, which can be expressed in terms of $W_{i-2}(t)$, and so on back to $V(t)$. This observation and little algebra leads to

PROPOSITION 2.1. *Let $\beta_{-1} = 0$, $\beta_i = \Pi_{j=0}^{i}\alpha_j$, $i = 0, \ldots, k$, and*

$$\tau(t) = t_0 + \beta_i(t - t_i) + \sum_{j=0}^{i-1} \beta_j(t_{j+1} - t), \; t \in [t_i, t_{i+1}), \; i = 0, \ldots, k-1.$$

Then $W(t) = V(\tau(t))$.

PROOF. $t \in [t_0, t_1) \Rightarrow W(t) = V(t_0 + \alpha_0[t - t_0]) = V(\tau(t))$

$$\begin{aligned} t \in [t_1, t_2) \Rightarrow W(t) = W_1(t) \;\; &= W_0(t_1 + \alpha_1(t - t_1)) \\ &= V(t_0 + \alpha_0[t_1 + \alpha_1(t - t_1) - t_0]) \\ &= V(t_0 + \beta_1[t - t_1] + \beta_0[t_1 - t_0]) = V(\tau(t)). \end{aligned}$$

Similarly, for $t \in [t_2, t_3)$

$$\begin{aligned} W(t) = \;\; & W_1(t_1 + \alpha_2(t - t_2)) \\ = \;\; & V(t_0 + \beta_1[t_2 + \alpha_2(t - t_2) - t_1] + \beta_0[t_1 - t_0]) \\ = \;\; & V(t_0 + \beta_2[t - t_2] + \beta_0[t_1 - t_0] + \beta_1[t_2 - t_1]) = V(\tau(t)). \end{aligned}$$

The general case follows by induction.

Note that α_i parameterizes the change in the degradation distribution from the interval $[t_{i-1}, t_i)$ to the interval $[t_i, t_{i+1})$, while $\beta_i = \Pi_{j=0}^{i}\alpha_j$ parameterizes the distribution of the degradation level of the $(i+1)^{st}$ interval $[t_i, t_{i+1})$. β_i is connected to the 'degradation rate' which is defined as follows.

DEFINITION 2.1. Let $W(t)$, $t \geq 0$, be a stochastic process for which $\frac{d}{dt}E(W(t))$ exists. Then we write $\delta(t) = \frac{d}{dt}E(W(t))$ and call $\delta(t)$ the *degradation* or *decay* rate for $W(t)$.

For the process $W(t) = V(\tau(t))$, it is clear that $\delta(t) = \eta\tau'(t) = \eta\beta_i$, $i \in [t_i, t_{i+1})$ and β_i is a multiple of the degradation rate in the interval $[t_i, t_{i+1})$. We give some examples of parameterizations.

EXAMPLE 2.1.

(a) $\alpha_i = \exp\{\theta(s_i - s_{i-1})\}$, $s_{i-1} \le s_i$. Note that when $s_{i-1} = s_i$, then $\alpha_i = 1$ and $\beta_{i-1} = \beta_i$. This should be the case since when the stress levels are the same, the degradation process parameters should be the same. For this α_i, β_i has the very simple form

$$\beta_i = \Pi_{j=0}^{i}\alpha_j = \exp\{\theta(s_i - s_0)\}.$$

Thus β_i only depends on the increase in stress from the first to the $(i+1)$st interval not on intermediate values of stress.

(b) $\alpha_i = (s_i/s_{i-1})^{\theta}$.
Again $\beta_{i-1} = \beta_i$ when $s_{i-1} = s_i$. Now

$$\beta_i = (s_i/s_0)^{\theta} = \exp\{\theta[\log s_i - \log s_0]\}.$$

Generally changing stress
Next we consider the case where the stress function $s(t)$ rather than being a step function with value s_i in the interval $[t_i, t_{i+1})$, can be an arbitrary non-negative (measurable) function. We again model the degradation process as

$$W(t) = V(\tau(t)), \quad r \ge t_0 \ge 0$$

where $\tau(t)$ is some function depending on $s(t)$ and t_0 is the starting point of the experiment.

An interesting special case is $t_0 = 0$. That is, we are starting the experiment with new units where the degradation level is $W(t_0) = V(0) = 0$. In this case $\tau(t_0) = \tau(0) = 0$.

PROPOSITION 2.2. *Suppose that $t_0 = 0$ and that $\tau(t)$ is continuous and increasing with $\tau(0) = 0$. If $V(t)$ is a Wiener process, and if T is the first time $V(\tau(t))$ crosses the boundary $\omega > 0$, then T has the distribution function $F_0(\tau(t))$, where F_0 is the inverse Gaussian distribution of Theorem 2.1.*

PROOF. See Doksum and Hóyland [7].

Note that if $\tau(t)$ is differentiable, with derivative $\beta(t)$, then the degradation in rate is

$$\delta(t) = \frac{d}{dt}EW(t) = \eta\beta(t).$$

Now if we take $\beta(s)$ to be a step function with value β_i on the interval $[t_i, t_{i+1})$, then $\tau(t)$ is exactly the function in Proposition 2.1 and we have obtained the step-stress accelerated failure time model as a special case of the current general framework. Conversely, we can think of the general case as a limit of the step-stress case with the widths of the intervals $[t_i, t_{i+1})$ tending to zero.

EXAMPLE 2.1 (continued).
Let $s(t)$ denote the continuous stress level at time t. Consider the step-stress model with

(a) $\alpha_i = \alpha(t_i) = \exp\{\theta[s(t_i) - s(t_{i-1})]\}$, $i = 1, \ldots, k$.
Then $\Pi_{j:t_j \leq t}\alpha(t_j)$ tends to $\exp\{\theta[s(t) - s(t_0)]\}$ as the widths of the intervals $[t_i, t_{i+1})$ tend to zero. Thus we get a general model with $\beta(t) = \exp\{\theta[s(t) - s(t_0)]\}$ and $\tau(t) = \int_0^t \beta(s)ds$.

(b) $\alpha_i = \alpha(t_i) = [s(t_i)/s(t_{i-1})]^\theta$.
Then $\Pi_{j:t_j \leq t}\alpha(t_j)$ tends to $[s(t)/s(t_0)]^\theta$ and we gat a general degradation process model $W(t) = V(\tau(t))$ with $\beta(t) = [s(t)/s(t_0)]^\theta$ and $\tau(t) = \int_0^t \beta(s)ds$.

Two recent books that treat accelerated life testing models different from the ones considered here are the books by Viertl [11] and Nelson [9].

2.2 *AIDS Research*

Next we turn to the HIV data. DeGruttola, Lange and Dafni [5], Lange, Carlin, and Gelfand [8], Berman [1], and Normand and Clearly [10], have considered experiments involving measurements on the immune system of people who have tested positive for the HIV virus. In particular, the data includes T4 cell counts at time points that are on the average 6 months apart for several hundred individuals. The ratios T4/T8 of T4 to T8 cells counts as well as covariate information, are also available. Now the process 1500-(T4 count at time t) can be regarded as a degradation process which can be modelled as $W(t) = V(t_0 + \alpha_0(t - t_0))$, $t \geq t_0$, where $V(t)$ is a Wiener process as before and t_0 and α_0 are parameters. This model implies that $E[W(t)] = \eta(t_0 + \alpha_0(t - t_0))$. DeGruttola, Lange and Dafni [5] and Lange, Carlin and Gelfand [8] have suggested that the mean T4 cell count decreases linearly with a certain slope until a change point c after which it decreases linearly with a steeper slope. In our Wiener process model this would correspond to $W(\tau(t))$ with

$$\begin{aligned} \tau(t) &= t_0 + \alpha_0(t - t_0), \quad t_0 \leq t < c \\ &= t_0 + \alpha_0\alpha_1[t - c] + \alpha_0[c - t_0], \quad t \geq c, \end{aligned}$$

where $\alpha_1 > 1$.

More generally, we can model the degradation level at time t as $V(\tau(t))$ where $V(t)$ is a Wiener process with drift η and diffusion constant σ^2. Without loss of generality we take $\eta = 1$ in which case $E(W(t)) = \tau(t)$ and the degradation rate is $\beta(t) = \tau'(t)$. In our change point example above we have

$$\begin{aligned} \beta(t) &= \alpha_0, \; t_0 \leq t < c \\ &= \alpha_0\alpha_1, \; t \geq c. \end{aligned}$$

Note that $\tau(t)$ involves t_0 which with the HIV data is not identifiable since it depends on the unknown infection time. On the other hand, α_0 and α_1 and the degradation rate $\beta(t)$ can be estimated from the data.

3 Hazard rate and degradation rate models

In this section we assume that T is a failure time and we are interested in estimating its failure distribution $F(t) = Pr(T \leq t)$. One approach is to frame parameters and questions in terms of the hazard rate $\lambda(t) = f(t)/[1 - F(t)]$ and the integrated hazard rate $\Lambda(t) = \int_0^t \lambda(s)ds$. Another approach, which is suggested in this section, is to think of the failure time T as the first time a degradation process $W(t)$ crosses a critical boundary $\omega > 0$. This leads to the degradation rate $\delta(t) = \frac{d}{dt}E(W(t))$ as a key concept. We consider the degradation model of Section 2 with $\eta = 1$, thus $\tau(t) = \int_0^t \delta(s)ds$.

There are analogies between hazard rate models and degradation rate models. Here are some.

(i) A general survival distribution $F(t)$ can be written as $F(t) = K(\Lambda(t))$ where $K(t) = 1-\exp(-t)$ is the standard exponential distribution and $\Lambda(t)$ is the integrated hazard rate. Similarly, any survival distribution $F(t)$ can be written in the form $F(t) = F_0(\tau(t))$ where F_0 is the standard inverse Gaussian distribution (the distribution of Theorem 2.1 with $\mu = 1$) and $\tau(t)$ is the integrated degradation rate.

(ii) A very useful form of $\Lambda(t)$ is βt^θ. This leads to the Weibull model where $F(t) = K(\beta t^\theta)$. Similarly, a linear stress level $s(t) = a + bt$ leads via Example 2.1(b) to a $\tau(t)$ of the form βt^θ and the failure distribution $F_0(\beta t^\theta)$.

(iii) Using the hazard rate $\lambda(t)$, it is easy to model the effect of covariates. Thus the Cox regression model has

$$\lambda(t) = \lambda_0(t)\ \exp\{\sum_{j=1}^{p} c_{ij}\beta_j\},$$

where $\lambda_0(t)$ is the baseline hazard rate (the hazard rate when the covariates have no effect), $\{c_{ij} : j = 1, \ldots, p\}$ are covariates for unit (or individual) i, and $\beta_1, \ldots, \beta_p$ are the Cox regression parameters. Similarly, we can model the degradation rate $\delta(t)$ using the semiparametric model

$$\delta(t) = \delta_0(t)\ \exp\{\sum_{j=1}^{p} c_{ij}\theta_j\},$$

where $\delta_0(t_0)$ is the baseline degradation rate and $\theta_1, \ldots, \theta_p$ are degradation rate regression parameters.

(iv) In the Cox regression model, the parameters $\beta_1, \ldots, \beta_p$ can be estimated using the partial or marginal likelihood. Similarly, in the semiparametric degradation model in (iii) preceding, the parameters $\theta_1, \ldots, \theta_p$ can be estimated using the partial or marginal likelihood as in Doksum [6] and Dabrowska and Doksum [4].

(v) Censored data can easily be accommodated using the hazard rate concept. This is also the case with the degradation rate concept. See Dabrowska and

Doksum [4] for semiparametric models and partial likelihood and Doksum and Hóyland [7] for parametric models and likelihood methods.

We end with a question: The Nelson-Aalen estimate is a very natural nonparametric estimate of $\Lambda(t)$; what is the 'natural' nonparametric estimate of $\tau(t)$?

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